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Necking in coating flow over periodic substrates

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Abstract The free-boundary problem determining the shape of a layer of viscous fluid coating a substrate while draining steadily under gravity is solved analytically for substrates taking the form of a periodic array of long plates of arbitrary width and spacing. The mathematical problem involves solving Poisson's equation with constant forcing term in the fluid layer subject to vanishing Neumann and Dirichlet conditions on the free boundary. By considering the problem in a potential plane and using conformal mapping, a two-parameter family of solutions is obtained in the form of an infinite series. Explicit, closed-form solutions are derived in the limiting cases of a single gap perforating an infinitely wide plate, and for an array of evenly spaced point plates. In these cases explicit expressions are obtained for the thinning, or necking, of the fluid layer in the gap regions between plates.

Keywords Coating · Free-boundary problem · Necking · Vortex layer

1 Introduction

The industrial coating process in which a cylindrical object, or substrate, is drawn vertically and parallel to its axis through a reservoir of viscous fluid is known as dip, or drag-out, coating; see e.g. [1–3]. Ignoring the effects of surface-tension and assuming steady flow, Tuck et al. [4] showed that far upstream of the reservoir, the shape of the coating layer is determined by a two-dimensional free-boundary problem in which, after suitable non-dimensionalisation, Poisson's equation $\nabla^2\psi = 1$ is solved within the fluid layer subject to ψ, and its first derivatives, vanishing on the free boundary. Here ψ is related to the along-substrate velocity field so that $\psi = \text{const}$ gives isotachs of this field. The boundary conditions correspond to zero stress at the free boundary and a demand that the mass flux in the fluid layer is maximised, this being justified on stability grounds [4]. On the substrate it is required that $\psi = C$, where the value of the constant C is related to the withdrawal speed of the substrate and effectively determines the maximum layer thickness. In the present study this constant is arbitrary, and solutions are normalised by insisting that the maximum layer thickness be unity. In industrial applications it is important to accurately predict the thinning of the coating layer near prominent geometric features of the substrate such as corners or gaps.

After their original formulation of the problem, Tuck et al. [4] went on to use a numerical method to compute the shape of the layer coating various substrates including a semi-infinite plate of zero-thickness and the corner of

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a right-angled block. Subsequently, Howison [5] showed that the semi-infinite plate problem can be solved exactly by considering the free-boundary problem in the potential plane. He showed that the thickness of the coating layer at the plate tip is diminished by a factor of $2/\pi$ compared to its upstream thickness. For geometries involving flows exterior and interior to right-angle corners, where methods based on the potential plane are unhelpful, Howison and King [6] used the method of Polubarinova-Kochina [7] to find exact solutions. The same method was subsequently used by Craster [8] to find exact solutions for the shapes of viscous layers coating wedge-shaped objects.

The equivalence of these coating solutions to fluid layers with uniform vorticity attached to substrates was pointed out by Crowdy [9]. In this interpretation the two-dimensional layer surrounding the substrate can be thought of as having uniform vorticity with an associated velocity field about the substrate which vanishes on the free boundary. In this case ψ plays the role of the stream function. Motivated by geophysical problems involving river discharges into coastal oceans and flows through narrow straits into larger oceans, Johnson and McDonald [10] used potential-plane techniques similar to that of Howison [5] to find exact steady solutions for fluid layers of uniform vorticity attached to an infinitely long wall driven by sources and sinks. More recently, the same authors [11] used similar techniques to find an explicit solution for steady flow with uniform vorticity around a finite plate of zero thickness, this being the first non-trivial example of such a solution for a body of finite extent. Equivalently, in the coating problem the solution [11] gives the shape of a layer of viscous fluid draining under gravity while in contact with a finite width plate. An explicit expression is obtained for the ratio of the thickness of the layer at the plate tips compared to its mid-plate value which shows that the factor of $2/\pi$ found by Howison [5] for a semi-infinite plate is a good approximation for finite plates of width greater than three times the maximum coating width.

In this paper an analytical solution taking the form of an infinite series is obtained for the layer coating a periodic substrate consisting of a zero-thickness plates of arbitrary width and spacing arranged in a straight line. Normalising so that the maximum coating width is unity, the solution to the general periodic problem depends on two parameters which, once specified, give the plate and gap widths along with the shape of the layer surrounding the plates. This problem has relevance to the coating of objects which have 'gaps' where it is important to quantify the relation in the gap width and the thinness of the coating layer in the necking region between the gap. Moreover, the solution of the present work generalises previously known solutions: the finite plate solution of [11] is simply a special case of the general solution obtained in the limit that the thickness in the necking region vanishes. Simplified, explicit solutions are obtained in the limiting cases of an infinitely long plate perforated by a single gap and for a periodic array of point plates.

2 Periodic plates

Figure 1a shows one period of a substrate in the form of a line of zero-thickness plates of length $2L$ with gaps between the plates of length $2W$. A complex coordinate system $z = x + iy$ is introduced such that the plates are aligned along $\mathfrak{Im} z = 0$ and $z = 0$ corresponds to the middle of a gap. The maximum thickness of the layer, with free boundary Γ, occurs at mid-plate and is normalised such that it has unit length. Let $0 < a < 1$ be the half-layer thickness at the neck $z = 0$, and let $b > 1$ be a parameter which is the value of $-\psi_y$ at $z = W + L$, i.e., at mid-plate. As $L \to \infty$, $b \to 1$ and as $L \to 0$, $b \to \infty$.

It is required to solve the free-boundary problem

$$\nabla^2\psi = 1, \;\; \psi = \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \Gamma, \quad \text{and} \quad \psi = C \text{ on the plate,} \tag{1}$$

where C is a constant. Let $-\psi$ be the imaginary part of an analytic function $f(z)$, and let $w = u - iv = f'(z)$. Then it follows that on the free boundary it is required $u = v = 0$, on the plate $v = 0$, and, in the coating layer, $v_x - u_y = 1$. In the vortical problem, $u - iv$ can be interpreted as the two-dimensional velocity field surrounding the plate, $v_x - u_y$ the vorticity and the parameter b is the value of u at mid-plate and is the minimum speed $|u|$ on the plate (see [11]). Also, in this interpretation the condition $\psi = C$ on the plate just reflects the fact that the boundary of the substrate is a streamline, whereas in the dip-coating application the value of C is related to the

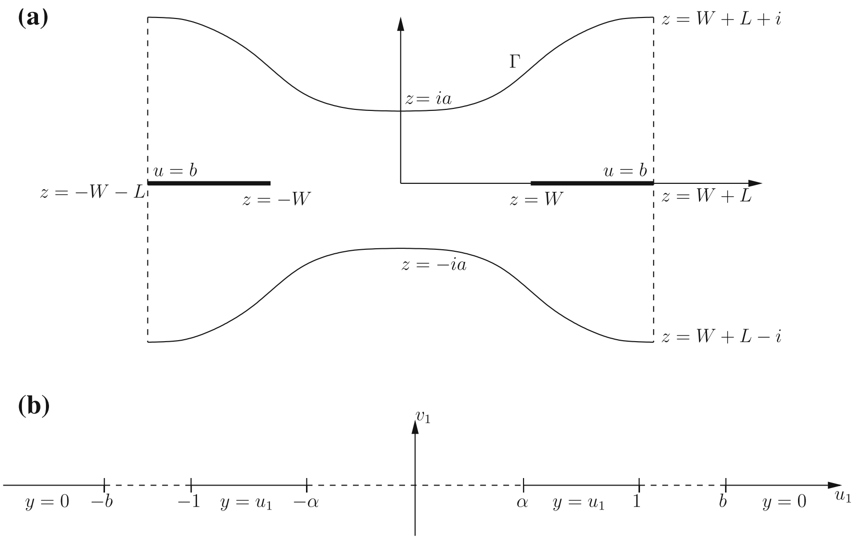

Fig. 1 **a** A sketch of the fluid layer and substrate over one period. The plates are of length $2L$ with distance $2W$ between them. The maximum layer thickness in unity and occurs mid-plate. The layer is symmetric about the middle of the gap. **b** The w_1-plane

withdrawal speed of the substrate and determines the thickness of the layer. In this work C is arbitrary and solutions are normalised by demanding that the maximum layer thickness is unity, as shown in Fig. 1a.

It is useful to write (see e.g. [9,11])

$$u - \mathrm{i}v = -\frac{\mathrm{i}}{2}(\bar{z} - S(z)), \tag{2}$$

where $S(z)$ is the (unique) Schwarz function of the free boundary, i.e., $\bar{z} = S(z)$ on Γ. Then (2) immediately gives $u - \mathrm{i}v = 0$ on Γ, as required. Moreover, the $-\mathrm{i}\bar{z}/2$ term in (2) implies $v_x - u_y = \nabla^2\psi = 1$ in the fluid layer. The solution must also satisfy the usual inverse-square-root singularity at the plate tips: $u - \mathrm{i}v \sim (z \pm W)^{-1/2}$ as $z \to \mp W$ (see e.g. [6]).

Defining the function

$$w_1 = u_1 + \mathrm{i}v_1 = -\frac{\mathrm{i}}{2}(z - S(z)), \tag{3}$$

it follows from (2) and (3) that $u_1 = u + y$ and $v_1 = -v$. Since $v = 0$ on both the free boundary Γ and the plates, they are mapped to the $\mathfrak{Re}w_1$-axis in the w_1-plane (i.e., the potential plane); see Fig. 1b. On Γ, $\mathfrak{Im}z = y = u_1$ and $a \le |u_1| \le 1$. On the plates, $\mathfrak{Im}z = y = 0$ and $b \le |u_1| < \infty$. These conditions on y are shown in Fig. 1b along with dashed lines which indicate where there is no condition on y.

Now consider the following sequence of mappings illustrated in Fig. 2. First, invoke the map $w_2 = w_1^2$ and note that along the positive $\mathfrak{Im}w_1$-axis the condition $y = 0$ holds, and, further, this axis maps to the negative $\mathfrak{Re}w_2$-axis. This is followed by an inversion $w_3 = w_2^{-1}$ and the translation $w_4 = w_3 - e_0$, where $e_0 = (a^{-2} + b^{-2} + 1)/3$. The three critical points on the $\mathfrak{Re}w_4$-axis become $e_1 = b^{-2} - e_0$, $e_2 = 1 - e_0$ and $e_3 = a^{-2} - e_0$, and have the property $e_1 + e_2 + e_3 = 0$. Finally, the Schwarz–Christoffel map

$$w_5 = \tfrac{1}{2}\int_{w_4}^{\infty} \frac{\mathrm{d}t}{\sqrt{(t - e_1)(t - e_2)(t - e_3)}} = \int_{w_4}^{\infty} (4t^3 - g_2 t - g_3)^{-\frac{1}{2}}\mathrm{d}t, \tag{4}$$

maps the $\mathfrak{Re}w_4$-axis to the boundary of the periodic rectangle in the w_5-plane. Note that (4) implies $w_4 = \wp(w_5)$, where $\wp$ is the Weierstrass $\wp$-function with invariants

$$g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3), \quad g_3 = 4e_1e_2e_3, \tag{5}$$

and half-periods ω and $\mathrm{i}\omega'$ (these periods being determined by g_2 and g_3; see e.g. [12]).

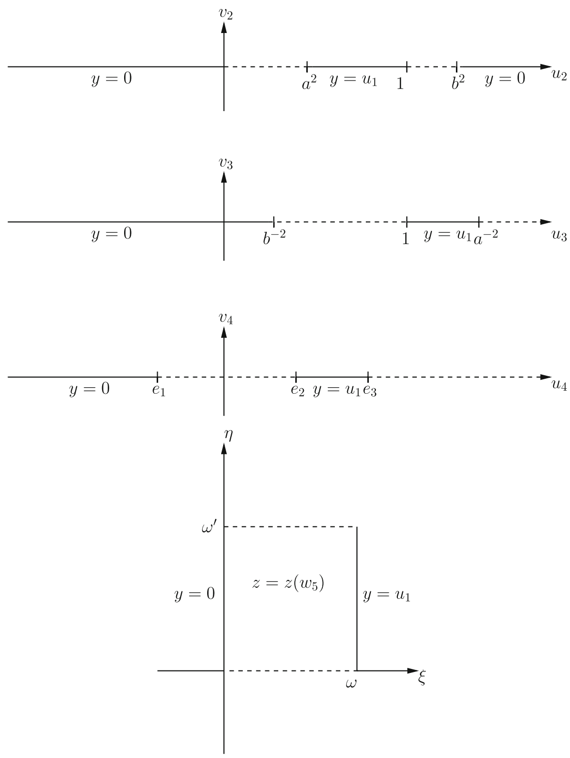

Fig. 2 Sequence of maps leading to periodic box with half-periods ω and ω' in the $w_5=\xi+\mathrm{i}\eta$ plane

Seeking a solution for z inside the w_5-box with period $2\omega'$ in the $\mathfrak{Im}w_5$-direction and for which y vanishes on $\mathfrak{Re}w_5=0$ gives

$$z=\mathrm{i}\frac{a_0 w_5}{2\omega}+\mathrm{i}\sum_{n=0}^{\infty}a_n\frac{\sinh\left(\frac{n\pi w_5}{\omega'}\right)}{\sinh\left(\frac{n\pi\omega}{\omega'}\right)}, \tag{6}$$

where $a_n,\ n=0,1,\ldots$ are real constants and $w_5=\xi+\mathrm{i}\eta$. On the free boundary $w_5=\omega+\mathrm{i}\eta$ and taking real and imaginary parts in (6) gives

$$x=-\frac{a_0\eta}{2\omega}-\sum_{n=0}^{\infty}a_n\coth\left(\frac{n\pi\omega}{\omega'}\right)\sin\left(\frac{n\pi\eta}{\omega'}\right), \tag{7}$$

$$y=\frac{a_0}{2}+\sum_{n=0}^{\infty}a_n\cos\left(\frac{n\pi\eta}{\omega'}\right). \tag{8}$$

The coefficients a_n are determined by noting on $w_5=\omega+\mathrm{i}\eta$

$$y(\eta)=\left[\wp(\omega+\mathrm{i}\eta)+e_0\right]^{-1/2}=\frac{a_0}{2}+\sum_{n=0}^{\infty}a_n\cos\left(\frac{n\pi\eta}{\omega'}\right), \tag{9}$$

and hence

$$a_n=\frac{2}{\omega'}\int_0^{\omega'}y(\eta)\cos\left(\frac{n\pi\eta}{\omega'}\right)\mathrm{d}\eta. \tag{10}$$

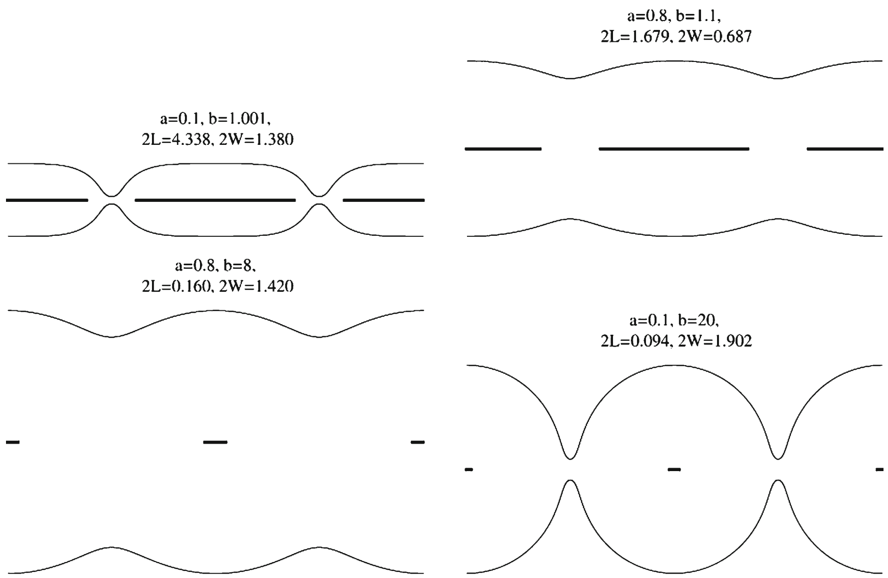

Fig. 3 Examples of coatings. In each case the maximum thickness of the layer is unity and occurs at mid-plate

Using the fact that at the plate tip, $w_1 \to \infty$ which, in turn, corresponds to $w_5 = \wp^{-1}(-e_0)$, and at the mid-point of the plate $w_1 = b$ which corresponds to $w_5 = \mathrm{i}\omega'$, it follows that half plate and gap lengths are

$$W = \frac{a_0 \gamma}{2\omega} + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi\gamma}{\omega'}\right), \tag{11}$$

$$L = W - \frac{a_0 \omega'}{2\omega}, \tag{12}$$

where $\gamma = -\mathrm{i}\wp^{-1}(-e_0)$. In summary, specifying a and b determines ω, ω' and e_0. Then a_n can be found using (10) and, finally, the layer shape is found from (7) and (8), with corresponding gap and plate widths given by (11) and (12).

Both the a_n and the sum in (7) can be evaluated rapidly by Fast Fourier Transform. The smoothness and periodicity of $y(\eta)$ means that the coefficients converge rapidly to zero and few terms are required to give highly accurate approximations to the solution.

Some example solutions of (7), (8) obtained using 1024 Fourier modes are shown in Fig. 3. The narrowing of the fluid layer in the gap region is referred to as 'necking'.

3 Single gap

In the limit $L \to \infty$ the plate geometry becomes that of an infinitely wide plate perforated by a single gap of width $2W$. In the same limit $b \to 1$ and the solution is characterised by the single parameter $0 < a < 1$, namely the half-thickness of the layer in the middle of the gap. In this case the potential plane problem shown in Fig. 1b can be mapped to the upper half of the w_2-plane using the conformal map

$$w_2 = u_2 + \mathrm{i}v_2 = \frac{w_1}{a} + \frac{\sqrt{w_1^2 - a^2}}{a}, \tag{13}$$

where the square root is taken to have positive imaginary part. Defining $\alpha = (1+\sqrt{1-a^2})/a$, the mathematical task is to find an analytic function $z = z(w_2)$ such that $\Im\mathfrak{m} z = y = u_1$ on $v_2 = 0$, $\alpha^{-1} < |u_2| < \alpha$ and $\Im\mathfrak{m} z = y = 0$

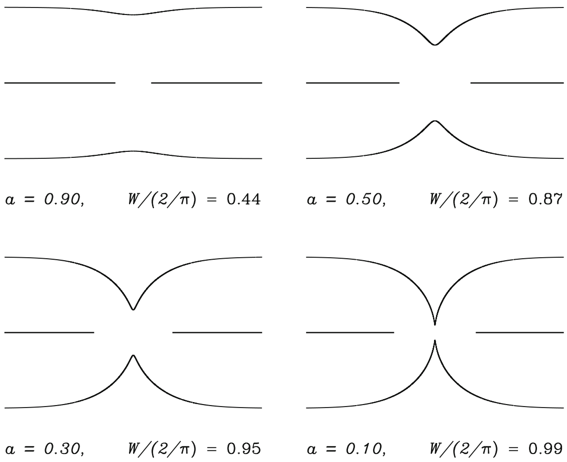

Fig. 4 Coating layer shapes for $a = 0.9, 0.5, 0.3$ and 0.1. Corresponding half gap-widths W are also given as a fraction of $2/\pi$

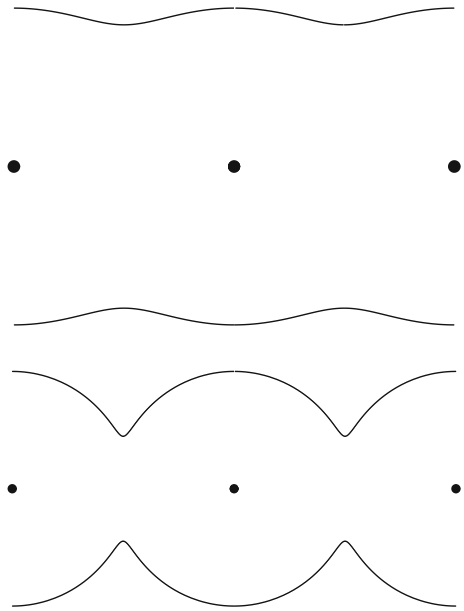

Fig. 5 The coating layer for point plates for (a) $a = 0.8,\ 2W = 1.39$ and (b) $a = 0.2,\ 2W = 1.89$

on $v_2 = 0$, $|u_2| < \alpha^{-1}$ and $|u_2| > \alpha$. Since $\mathfrak{Im} z(w_2)$ satisfies Laplace's equation, elementary methods give

$$z = \frac{w_1}{\pi} \log \left[\frac{(w_2 - \alpha)(w_2 + \alpha^{-1})}{(w_2 + \alpha)(w_2 - \alpha^{-1})} \right] + H(w_2), \tag{14}$$

where $H(w_2)$ is an analytic function whose imaginary part vanishes on $v_2 = 0$. Using the tip condition $z \pm W \to \mathrm{O}(w_1^{-1})$ as $z \to W$, asymptotic expansion of (14) in the limit $w_2 \to 0$ gives $H(w_2) \equiv 0$ and

$$W = \frac{2}{\pi}\sqrt{1 - a^2}. \tag{15}$$

On the free boundary $w_1 = y$ and $w_2 = t \in \mathbb{R}$, thus from (13) and (14) the coating layer can be expressed parametrically:

$$y = \frac{a}{2}(t + t^{-1}), \quad x = \frac{y}{\pi} \log \left[\frac{(\alpha - t)(\alpha t + 1)}{(\alpha + t)(\alpha t - 1)} \right], \tag{16}$$

where t is a real parameter such that $\alpha^{-1} < |t| < \alpha$.

Figure 4 gives examples of the layer shape in the vicinity of the gap for $a = 0.9, 0.5, 0.3$ and 0.1. From (15), $a \to 1$ as $W \to 0$ and as $a \to 0$, $W \to 2/\pi$. The latter limit describes the case of when the layers of the type described in [6] surrounding two semi-infinite plates just touch each other.

4 Periodic point plates

In the limit $L \ll 1$ the plates may be thought of as being 'point-like' and in the vortex-layer analogy the plates are replaced by point vortices with circulation equal and opposite to that of the vortex layer, since the velocity vanishes outside the vortex layer so the net circulation must be zero. In this limit, $b \to \infty$, and the corresponding potential plane problem is the appropriate limit of Fig. 1b. The problem can be solved by writing $z = \mathrm{i}w_1 + F(w_1)$ and

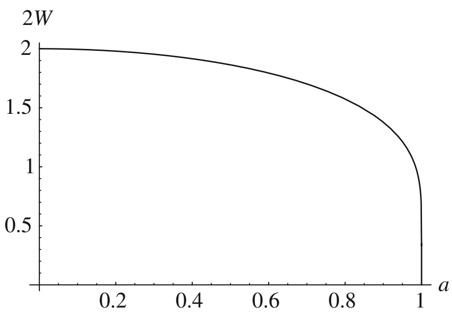

Fig. 6 Gap width $2W$ as a function of the necking thickness a for periodic point plates

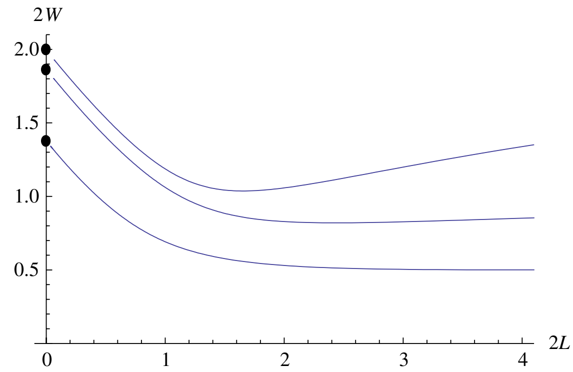

Fig. 7 Gap width $2W$ as a function of plate length $2L$ for $a = 0.9$ (lower curve), $a = 0.5$ (middle curve) and $a = 0.1$ (upper curve). The solid dots are the corresponding values of $2W$ given by the point-plate formula (19)

seeking an analytic function $F(w_1)$ such that $F \to \infty$ as $w_1 \to \infty$ and $\mathfrak{Im} F(w_1) = 0$ on $a \le |u_1| \le 1$. This is equivalent to finding the complex 'velocity potential' for uniform flow in the w_1-plane normal to two split 'plates' (or 'barriers') $a \le |u_1| \le 1$ along the real axis in the w_1-plane. Note, these are not the actual physical plates of the substrate, but artifacts of the potential plane map. The map $w_1 = a\,\mathrm{sn}\tau$, where sn is a Jacobian elliptic function, maps the w_1 plane to the periodic rectangle in the τ-plane (see e.g. [13]) and then using the results from [14], $F(w_1)$ can be found explicitly giving

$$z = \mathrm{i}\left[a\,\mathrm{sn}\tau + \zeta(\tau - \mathrm{i}K') + \zeta(\tau - 2K + \mathrm{i}K') + \zeta(2K)(1 - \tau/K)\right]. \tag{17}$$

Here

$$K(a) = \int_0^1 [(1-t^2)(1-a^2t^2)]^{-1/2}\mathrm{d}t, \quad K'(a) = K(\sqrt{1-a^2}), \tag{18}$$

are complete elliptic integrals and ζ is the Weierstrass zeta function with quasi-periods $4K$ and $2\mathrm{i}K'$ (see e.g. [12]). Figure 5 shows examples of the layer shape obtained by plotting real and imaginary parts of (17) for $a = 0.2$ and $a = 0.8$.

The gap width between the point plates can be found by putting $w_1 = 1$ in (17) and using the relation (18.3.37) of [15] giving

$$2W = \pi/K(a). \tag{19}$$

A plot of $2W$ versus a governed by (19) is shown in Fig. 6.

In the vortex layer interpretation, it follows from symmetry that the velocity field owing to the vortex layer vanishes at the points occupied by the 'plates'. Thus this solution also represents a new periodic vortex equilibrium solution in which the point plates correspond to point vortices embedded inside a layer of uniform vorticity such that there is zero net circulation. This equilibrium point vortex solution, like that of an infinite periodic array of point vortices (see e.g. [16]) is likely to be unstable if the point vortices are free to move (unlike the plates in the coating interpretation which are fixed).

5 Discussion

The thickness of the fluid layer in the necking region is of significance in industrial coating applications. Using the relations (11) and (12), Fig. 7 shows allowable plate and gap lengths required to achieve given neck thicknesses $a = 0.9$, 0.5 and 0.1. For example, if it is required that the minimum necking thickness a is 0.9 then W and L must be chosen so that they lie below the curve corresponding to $a = 0.9$. As the plate length $L \to 0$ the gap width W approaches that given by the point plate formula (19).

In addition to the infinite 'street' of point plates solution of Sect. 4, explicit solutions for the layer shape surrounding a *finite* collection of point plates can be obtained using Schwarz function techniques based on rational function maps $z = f(\zeta)$ from the unit circle in the ζ-plane. For example, if the point plates are arranged with N-fold symmetry, such solutions are closely related to the vortex equilibrium solutions found by Crowdy [17]. These, and solutions for other arrangements of a finite number of point plates, will be reported elsewhere. Additionally, there is a close relationship between the flows reported here and free-boundary flows in a Hele-Shaw cell. More precisely, the Baiocchi transform (see e.g. [18]) gives a direct relation between the stream function of the vortical flow problem and that of pressure in a Hele-Shaw cell, and implies that it is Laplace's equation rather than Poisson's equation which must be solved in the fluid interior. For example, in this analogy Fig. 5 represents merged blobs of fluid growing (going from (b) to (a)) from a linear array of point sources in a Hele-Shaw cell. In turn, the close analogy of Hele-Shaw flows with groundwater flow (e.g. [19]) and Stefan problems (e.g. [20]) suggest further useful analogies for the solutions given here.

Inspection of Figs. 3–5 shows that the curvature of the free boundary becomes increasingly large as a decreases. Thus it is inevitable that surface-tension effects will become important in this limit. However, with surface tension the potential plane mapping methods used here are no longer useful and it seems numerical methods must be used to find the shape of the coating layer. Nevertheless, the exact solutions presented will be of use in checking the results of such a numerical approach.

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